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New Statistical Results of Partial Sums of Ordered Gamma-Distributed RVs for Performance Evaluation of Wireless Communication Systems

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Abstract: With the increase in the complexity of communication systems, order statistics have become increasingly significant in performance evaluation, especially for advanced diversity techniques over fading channels. However, existing analytical approaches are not useful owing to their high complexity. In this paper, we present novel and insightful statistical results of the partial sums of ordered Gamma-distributed random variables. By adopting a unified analytical framework to obtain the desired joint probability density function from the joint moment-generating function, we redefine and derive the common key functions specialized for the Gamma distribution. The derived formula is presented in the form of either general closed-form expressions or at least a single integral form instead of the originally complicated multiple-integral form. In terms of a feasible application of our results, we present a complete and general closed-form analysis of the statistics of the combined received signal-to-noise ratio of the distributed cyclic delay diversity with the cyclic prefix single-carrier scheme. We also show that our analytical results can provide potential mathematical solutions for other wireless communication systems. Selected numerical examples are presented and discussed to illustrate the effectiveness of the applied approach and related results. All the derived analytical results were compared and verified by using Monte Carlo simulations to verify the accuracy of our analysis.

Keywords: order statistics; transmit diversity; combined SNR; distributed cyclic delay diversity; cyclic prefix single carrier

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1. Introduction

The accurate prediction of the performance of various emerging technologies can facilitate the adoption of the most suitable design choice in real-world systems. Hence, an analysis of their theoretical performance is valuable. Among the several mathematical and statistical tools used for analyzing the performance of digital wireless communication systems over fading channels, order statistics is an important subdiscipline of statistics theory [1,2]. The main objective of order statistics is to address the properties and distributions of ordered random variables (RVs) and their statistical functions. Order statistics is widely employed in several applications of statistical theory, such as life testing, quality control, signal and image processing, and the Internet of things [3–6]. In particular, in ordered RV-based statistical analysis, the most basic approach is to apply a multifold complex integration; however, the complexity increases significantly with an increase in the number of RVs considered for ordering. In this study, we show how recently obtained sophisticated results on order statistics can be applied to obtain elegant closed-form solutions. Our mathematical formalism is illustrated with an application to the required target joint statistics of the partial sums of ordered RVs over the Gamma distribution.

Transmit diversity techniques, such as maximal ratio transmission (MRT) [7,8] and distributed MRT (d-MRT) [9], are based on the assumption that the exact channel state information (CSI) is known at a transmitter. The signal-to-noise ratio (SNR) can be maximized by applying a transmit weight vector by using this information. Hence, for example, the diversity gains of MRT and d-MRT are proportional to the number of transmitting antennas and the number of cooperating single-antenna transmitters, respectively. A more practical transmit diversity scheme called cyclic delay diversity (CDD) has been considered to alleviate the challenging burden of CSI acquisition at the transmitter [10–14]. CDD has been widely adopted in practical wireless systems because of its compatibility with orthogonal frequency division multiplexing schemes [15–17]. Several previous studies [10–14] have dealt with cyclic prefix single-carrier (CP-SC) transmission [18,19], which is a good candidate for wireless systems involving cooperative relaying, spectrum sharing, and physical layer security [20–22].

However, distributed CDD (d-CDD) has not been studied, and the previous results on CDD CP-SC considered only an identical Rayleigh or identical frequency-selective fading channel. Therefore, the authors in [23,24] proposed a cooperative d-CDD CP-SC scheme over an independent but nonidentically distributed (i.n.i.d) frequency-selective fading channel. With this scheme, to maximize the combined received SNR, the best subset of single-antenna transmitters among the available cooperative transmitters is selected. Therefore, the use of order statistics is inevitable, which makes the mathematical performance analysis considerably more challenging. Although there have been several attempts to solve this problem, all previous analytical results regarding the d-CDD CP-SC scheme are marginal and mostly dependent on asymptotic analysis, even in independent and identically distributed (i.i.d.) channel environments. To the best of the authors' knowledge, a complete and general closed-form analysis of the statistics of the combined received SNR of the d-CDD CP-SC scheme is not sufficient to provide an overall mathematical framework.

1.1. Related Works

For performance evaluation, with the increase in the complexity of communication systems/algorithms, statistical analysis based on ordered RVs in a given distribution becomes increasingly important. However, this task is difficult compared with that in conventional communication systems. Note that, even if the original unordered RVs are independent, the ordered ones are dependent, owing to the inequality relations among them, making it challenging to analyze the joint statistics. It is much more difficult to obtain the target statistical result, that is, the probability density function (PDF) or moment generating function (MGF) of the partial sums of ordered RVs, compared with the sum of all ordered RVs or the best-ordered RVs. Therefore, an approximated or a simulation approach is often used [25–28]. To derive the exact closed-form expressions of the target statistics, several attempts have been made to convert dependent ordered RVs to independent unordered ones by applying the spacing method [29], the conditional PDF [30], and a successive conditioning approach [31]. However, these methods are applicable to only some simple special cases. In [32–34], although the statistical results for some types of partial sums considering all RVs were mentioned, the statistical results for partial sums of the best RVs have not been considered.

Fortunately, owing to the approach proposed in [35,36], where unified analytical frameworks were introduced to determine the required target joint statistics of the partial sums of ordered i.i.d. and i.n.i.d. RVs by extending the interesting results in [32–34], we can systematically derive the joint statistics of any case of the partial sums of ordered statistics in terms of the MGF and PDF, which are extremely important equations for performance analysis in communication theory.

All the previous results dealt with closed-form statistical results only for the Rayleigh distribution case. However, in this study, the Gamma distribution is considered, which is mathematically challenging. Note that the Gamma distribution is related to the Nakagami-*m* distribution in that both are viewed as a generalization of the Chi-squared distribution. Specifically, the Nakagami-*m* distribution exhibits a central Chi-squared distribution with $2m$ degrees of freedom [37]. This distribution is classically known as a versatile fading model because it is appropriate for modeling multipath fading in urban [38] and indoor [39] environments with a wide range of fading severity via the fading parameter, m, including the Rayleigh and Rice distributions. Owing to the significance of the performance analysis of the Nakagami-*m* (or Gamma) distribution, considerable research in this regard is still actively underway [27,40–44].

1.2. Contributions and Novelty

Although joint statistic results are essential for performance analysis in communication theory, the previous results are not valid for direct application to the model considered in this study. This is because they can only provide us with the analytical framework and not the required statistical results based on the statistical distribution in consideration, indicating that the problems of deriving the core equations for specific fading environments persist.

In this study, these unresolved problems are statistically analyzed by providing results in the form of either general closed-form expressions or at least a single integral form instead of the originally complicated multiple-integral form. If the number of RVs is large, even a single integral form is advantageous in terms of computational complexity compared with multifold integration. By adopting the analysis framework introduced in [35,36], we provide new statistical results that can be used for the performance analysis of wireless communication over fading channels of interest (e.g., Gamma distribution). The novelty of the results of this study lies in the newly derived common functions, which can be immediately applied to analyze the statistical performance of communication systems or algorithms based on any partial sum of ordered RVs over Gamma-distributed fading channels. By using these core equations, we demonstrate how to obtain the required target statistical results.

As we are not presenting a new scheme to improve the performance, our exact closed-form results are verified by demonstrating their consistency with simulation results. Then, our results can be used for accurate performance analysis without requiring simulation or approximation.

2. System and Channel Model

Following the same system model (this system is a good example of state-of-the-art technology to demonstrate our approach and contribution; our analysis, however, is not limited to it) in [23,24], we assumed M single-antenna transmitters and one single-antenna receiver. For the d-CDD-based CP-SC system, only $K(<M)$ CDD transmitters are selected for data transmission. By using known pilot symbols, this selection process is performed at the receiver by measuring each received SNR, denoted by $\gamma_k(1 \leq k \leq M)$, from the kth transmitter and then selecting the K largest ones.

Let S be the combined received SNR from the K selected transmitters; then, S is the sum of the K largest SNRs among M ones, that is, $S = \sum_{k=1}^{K} S_{k:M}$ where $S_{k:M}$ is the kth-order statistic (see [29] for terminology) such that $S_{1:M} \geq S_{2:M} \geq \cdots S_{K:M} \geq \cdots \geq S_{M:M}$. Based on the definition of S, we obtain

$$S = \sum_{k=1}^{K} S_{k:M} = \sum_{k=1}^{K} \mathbb{I}_k \gamma_k, \tag{1}$$

where $\mathbb{I}_k$ is the indication function given by

$$\Pr(\mathbb{I}_k = 1) = p_k \quad \text{and} \quad \Pr(\mathbb{I}_k = 0) = 1 - p_k. \tag{2}$$

Based on the Bernoulli process, the PDF of $S_{k:M}(= \mathbb{I}_k\gamma_k)$ is given by Equation (A.2) in [45]

$$f_{S_{k:M}}(x) = (1 - p_k)\delta(x) + p_k f_{\gamma_k}(x), \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta function.

As we consider distributed systems, different path losses are assumed in a large-scale fading model. As long as each channel experiences i.n.i.d frequency-selective fading, each γ_k is composed of a different number of multipath components. Hence, for a frequency-selective fading channel, the PDF of γ_k, $f_{\gamma_k}(\cdot)$ in (3), can be expressed as Equation (13) in [23]

$$f_{\gamma_k}(x) = \frac{x^{m_k-1}\exp\left(-\frac{x}{\eta_k}\right)}{\Gamma(m_k){\eta_k}^{m_k}}, \tag{4}$$

where m_k is the number of multipath components of the kth channel, $\Gamma(\cdot)$ is the complete gamma function [46] (Section (8.310)), and $\eta_k = \frac{P_T a_k}{\sigma_z^2}$ [23] where P_T is the transmission power of each transmitter, a_k is the path loss component of the kth channel, and σ_z^2 is the variance of noise. Note that, hereafter, for the mathematical tractability, we first consider the i.i.d. case where m_k and η_k in (4) do not depend on k.

3. Methodology: Statistics of the Combined Received SNR

In this section, we statistically analyze the combined received SNR given in (1). Specifically, our objective is to obtain meaningful analytical results regarding the sum of the K largest SNRs among M SNRs the PDF of which is given in (4).

If we let $Z_1 = \sum_{k=1}^{K-1} S_{k:M}$ and $Z_2 = S_{K:M}$, the target statistical result of $Z' = Z_1 + Z_2$ can be obtained from the two-dimensional joint PDF of Z_1 and Z_2, $f_Z(z_1, z_2)$, as

$$f_{Z'}(x) = \int_0^{\frac{x}{K}} f_Z(x - z_2, z_2)dz_2, \tag{5}$$

for $K \geq 2$. Note that, in general, the conventional MGF-based approach is commonly used for the sum of RVs to be applied to derive the target statistical results for the ordered-RV cases. Hence, the joint PDF, $f_Z(z_1, z_2)$, in (5) can be expressed by the inverse Laplace transform (LT) of the joint MGF as

$$f_Z(z_1, z_2) = \mathcal{L}^{-1}_{s_1,s_2}\{\mathrm{MGF}_Z(-s_1, -s_2)\}. \tag{6}$$

However, deriving this two-dimensional statistical result of Z_1 and Z_2 with the conventional MGF-based approach requires dealing with the computationally intensive and complex original statistical result, the MGF expression of which involves a multiple integral (i.e., K-fold integrals in this case), as shown in the following equation:

$$\begin{aligned}
\mathrm{MGF}_Z(\lambda_1, \lambda_2) &= E\{\exp(\lambda_1 z_1 + \lambda_2 z_2)\} \\
&= F\int_0^{\infty} dS_{1:K} f_{S_{1:K}}(S_{1:K})\exp(\lambda_1 S_{1:K})\times \\
&\qquad \vdots \\
&\quad \times\int_0^{S_{K-2:M}} dS_{K-1:M} f_{S_{K-1:M}}(S_{K-1:M})\exp(\lambda_1 S_{K-1:M}) \\
&\quad \times\int_0^{S_{K-1:M}} dS_{K:M} f_{S_{K:M}}(S_{K:M})\exp(\lambda_2 S_{K:M})[c(S_{K:M})]^{M-K},
\end{aligned} \tag{7}$$

where $F = \frac{M!}{(M-K)!}$, and $c(S_{K:M})$ is the cumulative distribution function of $S_{K:M}$ given by

$$c(S_{K:M}) = \int_0^{S_{K:M}} dx f_{S_{K:M}}(x). \tag{8}$$

In this study, by adopting the analytical framework proposed in [35], we obtain a target two-dimensional joint PDF of the arbitrary partial sums of ordered RVs, for example, involving the first K ordered RVs among M ones, as a closed-form result. The first step is to derive the analytical expressions of the two-dimensional joint MGF of the partial sums. Then, in step two, we apply the inverse LT to obtain the target two-dimensional joint PDF.

Through step one, following the analytical framework proposed in [35], we can obtain a two-dimensional joint MGF of $Z = [Z_1, Z_2]$ in a single integral expression given by

$$\begin{aligned} \mathrm{MGF}_Z(\lambda_1, \lambda_2) =& \frac{F}{(K-1)!} \int_0^\infty dS_{K:M} f_{S_{K:M}}(S_{K:M}) \\ & \times \exp(\lambda_2 S_{K:M}) [c(S_{K:M})]^{M-K} [e(S_{K:M}, \lambda_1)]^{K-1}, \end{aligned} \tag{9}$$

where $e(S_{K:M}, \lambda_1)$ is the mixture of an exceedance distribution function and an MGF defined in [35] and can be obtained as

$$e(S_{K:M}, \lambda_1) = \int_{S_{K:M}}^\infty dx f_{S_{K:M}}(x) \exp(\lambda_1 x). \tag{10}$$

Substituting (8) and (10) into (9) and then applying the inverse LT, the target two-dimensional joint PDF can be expressed as

$$\begin{aligned} f_Z(z_1, z_2) &= \mathcal{L}^{-1}_{s_1, s_2}\{\mathrm{MGF}_Z(-s_1, -s_2)\} \\ &= \frac{F}{(K-1)!} f_{Z_2}(z_2) [c(z_2)]^{M-K} \mathcal{L}^{-1}_{s_1}\left\{ [e(z_2, -s_1)]^{K-1} \right\}, \end{aligned} \tag{11}$$

where $f_{Z_2}(z_2)$ is expressed by (3). Gamma-distributed RVs, based on (4), (8) and (10) can be expressed by

$$\begin{aligned} c(S_{K:M}) &= \int_0^{S_{K:M}} dx f_{S_{K:M}}(x) \\ &= \frac{p_k}{\Gamma(m_k)} \left[\Gamma(m_k) - \Gamma\left(m_k, \frac{S_{K:M}}{\eta_k}\right) \right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} e(S_{K:M}, \lambda_1) &= \int_{S_{K:M}}^\infty dx f_{S_{K:M}}(x) \exp(\lambda_1 x) \\ &= \frac{p_k}{\Gamma(m_k)} \left(\frac{\eta_k}{S_{K:M}} \right)^{-m_k} \mathrm{E}_{1-m_k}\left(S_{K:M}\left(-\lambda_1 + \frac{1}{\eta_k} \right) \right) \\ &\overset{\text{or}}{=} \frac{p_k}{\Gamma(m_k)} \left(\frac{\eta_k}{S_{K:M}} \right)^{-m_k} \left(S_{K:M}\left(-\lambda_1 + \frac{1}{\eta_k} \right) \right)^{-m_k} \Gamma\left(m_k, S_{K:M}\left(-\lambda_1 + \frac{1}{\eta_k} \right) \right) \\ &= \frac{p_k}{\Gamma(m_k)} \eta_k^{-m_k} \left(-\lambda_1 + \frac{1}{\eta_k} \right)^{-m_k} \Gamma\left(m_k, S_{K:M}\left(-\lambda_1 + \frac{1}{\eta_k} \right) \right), \end{aligned} \tag{13}$$

respectively, where $\Gamma(\cdot, \cdot)$ denotes an incomplete gamma function [46] (Section (8.350)), and $\mathrm{E}_n(x) = x^{n-1}\Gamma(1-n, x)$ denotes an exponential integral function [46] (Section (3.381)) [47]. Notably, $[c(z_2)]^{M-K}$ and $[e(z_2, -s_1)]^{K-1}$ in (11) need to be derived in an integrable form. Hence, based on the derivation in Appendix A, $[c(z_2)]^{M-K}$ in (11) can be obtained as (14).

By using a similar approach, $[e(z_2, -s_1)]^{K-1}$ can be expressed in the summation form given in (15). Finally, for $z_1 > (K-1)z_2$, we can obtain $\mathcal{L}^{-1}_{s_1}\left\{ [e(z_2, -s_1)]^{K-1} \right\}$ in (11) as (16), and the detailed derivations are presented in Appendix B.

$$
\begin{aligned}
[c(z_2)]^{M-K} &= \sum_{l=0}^{M-K} \binom{M-K}{l} \frac{(-1)^l p_k^{M-K}}{(\Gamma(m_k))^l} [(m_k-1)!]^l \exp\left(-\frac{l}{\eta_k} z_2\right) \\
&\times \sum_{\substack{i_0, i_1, \cdots, i_{m_k-1} \geq 0 \\ i_0+i_1+\cdots+i_{m_k-1}=l}} \frac{l!}{i_0! i_1! \cdots i_{m_k-1}!} \prod_{j=0}^{m_k-1} \left(\frac{1}{j!}\right)^{i_j} \left(\frac{z_2}{\eta_k}\right)^{\sum_{j=0}^{m_k-1} j \cdot i_j} \\
&\overset{\text{or}}{=} \sum_{l=0}^{M-K} \binom{M-K}{l} \frac{(-1)^l p_k^{M-K}}{(\Gamma(m_k))^l} [(m_k-1)!]^l \exp\left(-\frac{l}{\eta_k} z_2\right) \\
&\times \sum_{i_1=0}^{m_k-1} \sum_{i_2=0}^{m_k-1} \cdots \sum_{i_l=0}^{m_k-1} \frac{1}{\prod_{j=1}^{l} i_j!} \left(\frac{z_2}{\eta_k}\right)^{\sum_{j=1}^{l} i_j}.
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
[e(z_2,-s_1)]^{K-1} &= \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1} \eta_k^{-(K-1)m_k} [(m_k-1)!]^{K-1} \left(s_1+\frac{1}{\eta_k}\right)^{-(K-1)m_k} \exp\left(-(K-1)\left(s_1+\frac{1}{\eta_k}\right)z_2\right) \\
&\times \sum_{\substack{i_0, i_1, \cdots, i_{m_k-1} \geq 0 \\ i_0+i_1+\cdots+i_{m_k-1}=K-1}} \frac{(K-1)!}{i_0! i_1! \cdots i_{m_k-1}!} \prod_{j=0}^{m_k-1} \left(\frac{1}{j!}\right)^{i_j} \left(\left(s_1+\frac{1}{\eta_k}\right)z_2\right)^{\sum_{j=0}^{m_k-1} j \cdot i_j} \\
&\overset{\text{or}}{=} \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1} \eta_k^{-(K-1)m_k} [(m_k-1)!]^{K-1} \left(s_1+\frac{1}{\eta_k}\right)^{-(K-1)m_k} \exp\left(-(K-1)\left(s_1+\frac{1}{\eta_k}\right)z_2\right) \\
&\times \sum_{i_1=0}^{m_k-1} \sum_{i_2=0}^{m_k-1} \cdots \sum_{i_{K-1}=0}^{m_k-1} \frac{1}{\prod_{j=1}^{K-1} i_j!} \left(\left(s_1+\frac{1}{\eta_k}\right)z_2\right)^{\sum_{j=1}^{K-1} i_j}.
\end{aligned}
\tag{15}
$$

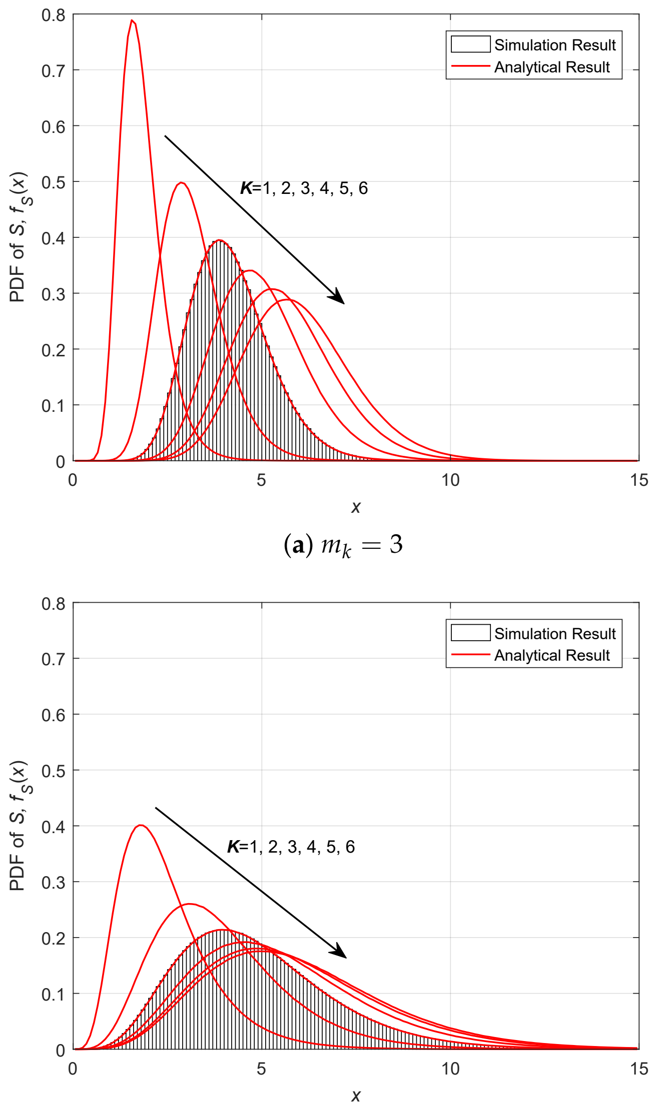
$$
\begin{aligned}
\mathcal{L}_{s_1}^{-1}\left\{[e(z_2,-s_1)]^{K-1}\right\} &= \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1} \left(\frac{1}{\eta_k}\right)^{(K-1)m_k} [(m_k-1)!]^{K-1} \\
&\times \sum_{\substack{i_0, i_1, \cdots, i_{m_k-1} \geq 0 \\ i_0+i_1+\cdots+i_{m_k-1}=K-1}} \frac{(K-1)!}{i_0! i_1! \cdots i_{m_k-1}!} \prod_{j=0}^{m_k-1} \left(\frac{1}{j!}\right)^{i_j} \frac{1}{\Gamma\left((K-1)m_k - \sum_{j=0}^{m_k-1} j \cdot i_j\right)} \\
&\times z_2^{\sum_{j=0}^{m_k-1} j \cdot i_j} (z_1-(K-1)z_2)^{(K-1)m_k - \sum_{j=0}^{m_k-1} j \cdot i_j - 1} \exp\left(\frac{1}{\eta_k} z_1\right) \\
&\overset{\text{or}}{=} \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1} \left(\frac{1}{\eta_k}\right)^{(K-1)m_k} [(m_k-1)!]^{K-1} \\
&\times \sum_{i_1=0}^{m_k-1} \sum_{i_2=0}^{m_k-1} \cdots \sum_{i_{K-1}=0}^{m_k-1} \frac{1}{\prod_{j=1}^{K-1} i_j!} \cdot \frac{1}{\Gamma\left((K-1)m_k - \sum_{j=1}^{K-1} i_j\right)} \\
&\times z_2^{\sum_{j=1}^{K-1} i_j} (z_1-(K-1)z_2)^{(K-1)m_k - \sum_{j=1}^{K-1} i_j - 1} \exp\left(\frac{1}{\eta_k} z_1\right).
\end{aligned}
\tag{16}
$$

4. Numerical Results and Discussion

In this section, we apply the closed-form results from the previous section to the statistics of the combined received SNR given in (1) for the d-CDD-based CP-SC system. Note that the PDF of γ_k in (4) follows the central Chi-squared distribution with $2m_k$ degrees of freedom; therefore, the average SNR of γ_k, $\overline{\gamma}$, can be obtained as $\eta_k m_k$.

Figure 1 shows the PDF of S, $f_S(x)$, for various values of K when $M = 6$ and $\overline{\gamma} = 1$. All the analytical results derived in this study were compared and verified by using Monte Carlo simulations, e.g., refer to the perfect match when $K = 3$. Figure 1b shows a special case of $m_k = 1$ corresponding to the exponential distribution, that is, the Rayleigh distribution. It can be further confirmed that, when $M = K = 6$, the results in Figure 1a,b are consistent with the well-known maximal ratio combined in Nakagami and Rayleigh fading environments [48,49], respectively, and when $K = 1$, selection combined in Nakagami and Rayleigh fading environments [48,49], respectively. Figure 2 shows the effects of m_k on the statistics of the combined received SNR when $M = 6$, $K = 3$, and $\overline{\gamma} = 1$. Similarly, all the analytical results derived in this study were compared and verified by using Monte Carlo simulations, e.g., refer to the exact match when $m_k = 2$.

Note that common functions or core equations must be obtained to derive the final results by applying the proposed approach. This is a major limitation of this study. However, if it is difficult to obtain the core equations in closed form, similar closed-form results cannot be obtained even if other existing methods are applied. In any circumstance, our approach provides considerably simpler solutions than the conventional method involving multifold integration.



(**a**) $m_k = 3$

(**b**) $m_k = 1$

Figure 1. PDF of S, $f_S(x)$, for various values of K when $M = 6$ and $\overline{\gamma} = 1$.

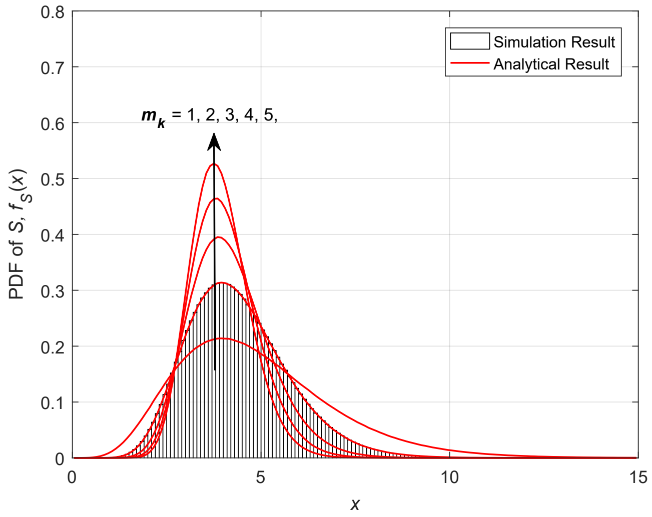


Figure 2. PDF of S, $f_S(x)$, for various values of m_k when $M = 6$, $K = 3$, and $\overline{\gamma} = 1$.

Moreover, the process of obtaining the resulting statistics and core functions in this study can find applications in several research problems in wireless communication systems that deal with the partial sums of ordered RVs. Specifically, based on the results of this study, we can expand the scope of previously limited analysis to more versatile Gamma-fading environments. Therefore, various research topics, such as advanced diversity combining [29,30], channel adaptive transmission [31], multiuser scheduling [50], and advanced RAKE receiver [51–54], can be sophisticatedly analyzed over the Gamma distribution.

5. Conclusions and Recommendations

Order statistics has become increasingly significant in the performance analysis of wireless communication systems based on advanced diversity techniques over fading channels. These advanced techniques require novel and more complicated order statistical results. However, existing analytical approaches are not useful owing to their high complexity. For example, in the recently proposed d-CDD, the use of order statistics was inevitable, but the related previous analytical results mostly relied on asymptotic analysis, even in the case of i.i.d. channel environments. In this study, by deriving the new key common functions necessary for the analysis, we provided novel and insightful statistical results of the partial sums of ordered Gamma-distributed RVs. We performed detailed mathematical manipulations and introduced new representations to obtain generic results (e.g., joint MGF and related joint PDF) for i.i.d. Gamma distribution in a compact form. In this regard, we showed that our results are useful for obtaining the required analytical results of the combined received SNR of the d-CDD CP-SC scheme. In the future, we will analyze the i.n.i.d. case to consider more general and versatile fading environments. The main challenge in generalizing the results of this study to i.n.i.d. general fading cases is that the joint PDF of ordered i.n.i.d. RVs are considerably more complicated than ordered i.i.d. RVs.

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Appendix A. Derivations of (14) and (15)

Based on (8), with binomial expansion and mathematical simplification, $[c(z_2)]^{M-K}$ in (11) can be expressed as

$$\begin{aligned}[c(z_2)]^{M-K} &= \left(\frac{p_k}{\Gamma(m_k)}\right)^{M-K}\left[\Gamma(m_k)-\Gamma\left(m_k,\frac{z_2}{\eta_k}\right)\right]^{M-K} \\ &= \sum_{l=0}^{M-K}\binom{M-K}{l}\frac{(-1)^l p_k^{M-K}}{(\Gamma(m_k))^l}\left(\Gamma\left(m_k,\frac{z_2}{\eta_k}\right)\right)^l.\end{aligned} \tag{A1}$$

In (A1), the kth power of $\Gamma(n,x)$ for a non-negative integer k can be rewritten as [55]

$$[\Gamma(n,x)]^k = [(n-1)!]^k \exp(-kx)\left[\sum_{m=0}^{n-1}\frac{1}{m!}x^m\right]^k. \tag{A2}$$

Note that, based on [56] (Equation (9)), $\left[\sum_{m=0}^{n-1}\frac{1}{m!}x^m\right]^k$ in (A2) can be reformulated from the multiple-product form to the summation form as follows:

$$\begin{aligned}\left[\sum_{m=0}^{n-1}\frac{1}{m!}x^m\right]^k &= \sum_{\substack{i_0,i_1,\cdots,i_{n-1}\geq 0\\ i_0+i_1+\cdots+i_{n-1}=k}}\frac{k!}{i_0!i_1!\cdots i_{n-1}!}\prod_{m=0}^{n-1}\left(\frac{1}{m!}x^m\right)^{i_m} \\ &= \sum_{\substack{i_0,i_1,\cdots,i_{n-1}\geq 0\\ i_0+i_1+\cdots+i_{n-1}=k}}\frac{k!}{i_0!i_1!\cdots i_{n-1}!}\prod_{m=0}^{n-1}\left(\frac{1}{m!}\right)^{i_m} x^{\sum_{m=0}^{n-1} m\cdot i_m} \\ &\overset{\text{or}}{=} \sum_{m_1=0}^{n-1}\sum_{m_2=0}^{n-1}\cdots\sum_{m_k=0}^{n-1}\frac{x^{m_1+m_2+\cdots+m_k}}{m_1!m_2!\cdots m_k!} \\ &= \sum_{m_1=0}^{n-1}\sum_{m_2=0}^{n-1}\cdots\sum_{m_k=0}^{n-1}\frac{x^{\sum_{i=1}^k m_i}}{\prod_{i=1}^k m_i!},\end{aligned} \tag{A3}$$

which is appropriate for the integral operation. Hence, we obtain the following form:

$$\begin{aligned}\left[\Gamma\left(m_k,\frac{z_2}{\eta_k}\right)\right]^l &= [(m_k-1)!]^l \exp\left(-l\frac{z_2}{\eta_k}\right) \\ &\times \sum_{\substack{i_0,i_1,\cdots,i_{m_k-1}\geq 0\\ i_0+i_1+\cdots+i_{m_k-1}=l}}\frac{l!}{i_0!i_1!\cdots i_{m_k-1}!}\prod_{j=0}^{m_k-1}\left(\frac{1}{j!}\right)^{i_j}\left(\frac{z_2}{\eta_k}\right)^{j\cdot i_j} \\ &\overset{\text{or}}{=}[(m_k-1)!]^l \exp\left(-l\frac{z_2}{\eta_k}\right) \\ &\times \sum_{i_1=0}^{m_k-1}\sum_{i_2=0}^{m_k-1}\cdots\sum_{i_l=0}^{m_k-1}\frac{1}{\prod_{j=1}^l i_j!}\left(\frac{z_2}{\eta_k}\right)^{\sum_{j=1}^l i_j},\end{aligned} \tag{A4}$$

which yields (14). By using the same property as (A3), we can also obtain $[e(z_2,-s_1)]^{K-1}$ in the summation form, as shown in (15).

Appendix B. Derivations of (16)

The inverse LT of $[e(z_2, -s_1)]^{K-1}$ can be expressed for $z_1 > (K-1)z_2$ as

$$\begin{aligned}
\mathcal{L}_{s_1}^{-1}\left\{[e(z_2,-s_1)]^{K-1}\right\} &= \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1}\left(\frac{1}{\eta_k}\right)^{(K-1)m_k}[(m_k-1)!]^{K-1} \\
&\times \sum_{\substack{i_0,i_1,\cdots,i_{m_k-1}\geq 0 \\ i_0+i_1+\cdots+i_{m_k-1}=K-1}} \frac{(K-1)!}{i_0!i_1!\cdots i_{m_k-1}!}\prod_{j=0}^{m_k-1}\left(\frac{1}{j!}\right)^{i_j} \\
&\times \mathcal{L}_{s_1}^{-1}\left\{\frac{1}{\left(s_1+\frac{1}{\eta_k}\right)^{(K-1)m_k}}\cdot\left(\left(s_1+\frac{1}{\eta_k}\right)z_2\right)^{\sum_{j=0}^{m_k-1} j\cdot i_j}\right. \\
&\times \left.\exp\left(-(K-1)\left(s_1+\frac{1}{\eta_k}\right)z_2\right)\right\} \\
&\overset{\text{or}}{=} \left[\frac{p_k}{\Gamma(m_k)}\right]^{K-1}\left(\frac{1}{\eta_k}\right)^{(K-1)m_k}[(m_k-1)!]^{K-1} \\
&\times \sum_{i_1=0}^{m_k-1}\sum_{i_2=0}^{m_k-1}\cdots\sum_{i_{K-1}=0}^{m_k-1}\frac{1}{\prod_{j=1}^{K-1} i_j!} \\
&\times \mathcal{L}_{s_1}^{-1}\left\{\frac{1}{\left(s_1+\frac{1}{\eta_k}\right)^{(K-1)m_k}}\cdot\left(\left(s_1+\frac{1}{\eta_k}\right)z_2\right)^{\sum_{j=1}^{K-1} i_j}\right. \\
&\times \left.\exp\left(-(K-1)\left(s_1+\frac{1}{\eta_k}\right)z_2\right)\right\},
\end{aligned} \tag{A5}$$

where $(K-1)m_k > \sum_{j=0}^{m_k-1} j \cdot i_j$. In (A5), the inverse LT term can be rewritten as

$$z_2^{\sum_{j=0}^{m_k-1} j\cdot i_j}\mathcal{L}_{s_1}^{-1}\left\{\frac{1}{\left(s_1+\frac{1}{\eta_k}\right)^{(K-1)m_k-\sum_{j=0}^{m_k-1} j\cdot i_j}}\exp\left(-(K-1)\left(s_1+\frac{1}{\eta_k}\right)z_2\right)\right\}. \tag{A6}$$

By applying the inverse LT pair (i.e., $t^n \longleftrightarrow n!/s^{n+1}$) and properties (i.e., frequency and time shifts), we can solve (A6) as

$$z_2^{\sum_{j=0}^{m_k-1} j\cdot i_j}\frac{1}{\Gamma\left((K-1)m_k-\sum_{j=0}^{m_k-1} j\cdot i_j\right)}(z_1-(K-1)z_2)^{(K-1)m_k-\sum_{j=0}^{m_k-1} j\cdot i_j - 1}\exp\left(\frac{1}{\eta_k}z_1\right). \tag{A7}$$

Finally, substituting (A7) into (A5) yields (16).

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